Discontinuous Galerkin Discretisation of the Buoyancy Balance Equation

Chris Pain, Matthew Piggott, David Ham*et al*

March 2, 2007

Since the buoyancy term is a very large part of the pressure for many of our problems constituted by the buoyancy term. Since this term

Consider a component of pressure which results from the buoyancy term:

$$\frac{\partial p_{\mathrm{b}}}{\partial z} = -\rho g \tag{1}$$

This expression is valid for a flat Earth with a constant gravity vector. On a spheroidal Earth the gravity vector $\mathbf{g}$ points towards the centre of the Earth, a direction which is not in general aligned with the z-axis. Now if we write $\mathbf{q}$ for the local upward unit vector we can reformulate this equation on the sphere as:

$$\mathbf{q} \cdot \nabla p_{\mathrm{b}} = \mathbf{q} \cdot \rho \mathbf{g} \tag{2}$$

If we now write M_i for some space of test functions and integrate over an element Ω_e with boundary Γ_e we can produce a weak form in the conventional way:

$$\int_{\Omega_e} M_i \mathbf{q} \cdot \nabla p_{\mathrm{b}} \mathrm{d}V = \int_{\Omega_e} M_i \mathbf{q} \cdot \rho \mathbf{g} \mathrm{d}V \tag{3}$$

Assume for the moment that $\rho\mathbf{g}$ is represented in some continuous trial space. Then we focus on the discontinous discretisation of the left hand side. Integrating by parts twice we have

$$\int_{\Omega_e} M_i \mathbf{q} \cdot \nabla p_{\mathrm{b}} \mathrm{d}V = \int_{\Omega_e} M_i \mathbf{q} \cdot \nabla p_{\mathrm{b}} \mathrm{d}V - \int_{\Gamma_e} M_i \mathbf{q} \cdot \mathbf{n} [p_{\mathrm{b}}] \mathrm{d}\Gamma \tag{4}$$

where $\mathbf{n}$ is the element boundary outward normal and $[\cdot]$ indicates the jump across the element boundary. Various choices of jump condition are possible. The centred average is:

$$\begin{aligned} [p_{\mathrm{b}}] &= p_{\mathrm{b}}^{\mathrm{int}} - \frac{1}{2}(p_{\mathrm{b}}^{\mathrm{int}} + p_{\mathrm{b}}^{\mathrm{ext}}) \\ &= \frac{1}{2}(p_{\mathrm{b}}^{\mathrm{int}} - p_{\mathrm{b}}^{\mathrm{ext}}) \end{aligned} \tag{5}$$

where $p_{\mathrm{b}}^{\mathrm{int}}$ is the value of p_{b} in this element at the boundary and $p_{\mathrm{b}}^{\mathrm{ext}}$ is the value in the adjacent element or on the domain boundary as appropriate. One might also consider $-\mathbf{q}$ as an advecting direction and calculate an upwinded jump condition:

$$[p_{\mathrm{b}}] = (p_{\mathrm{b}}^{\mathrm{int}} - p_{\mathrm{b}}^{\mathrm{upw}}) \tag{6}$$

where upw is the value of the p_{b} on the upper side of the face. Note that this expression is zero at downwind boundaries.

In equation (4), we choose the upwinded method which reflects the direction of information propagation in the buoyancy term.

Having found p_{b} we in this way, we now move to discretise the buoyancy and buoyancy pressure terms in the momentum equation:

$$-\nabla p_{\mathrm{b}} + \rho \mathbf{g} \tag{7}$$

It will be useful to consider this expression in terms of its vertical and horizontal components. Once again we encounter the complications presented by the spherical Earth representation. The objective is to express this term in a manner which shows that vertical component exactly cancels.

Define a rotation matrix R_{v} which maps $(0, 0, 1)$ to $\mathbf{q}$:

$$R_{\mathrm{v}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ q_x & q_y & q_z \end{bmatrix} \tag{8}$$

by choosing an orthonormal basis orthogonal to $\mathbf{q}$ we can likewise construct a rotation matrix to extract the horizontal terms:

$$R_{\mathrm{h}} = \begin{bmatrix} t_{1x} & t_{1y} & t_{1z} \\ t_{2x} & t_{2y} & t_{2z} \\ 0 & 0 & 0 \end{bmatrix} \tag{9}$$

The full rotation matrix $R = R_{\mathrm{v}} + R_{\mathrm{h}}$ then maps the local orientation to a space in which $(0, 0, 1)$ is up. Note than since R is a unitary matrix $R^{\mathrm{T}} R = I$.

If we produce a weak formulation of (7) with respect to the momentum basis functions N_i and integrate by parts twice we have:

$$\int_{\Omega_e} N_i \left(-\nabla p_{\mathrm{b}} + \rho \mathbf{g}\right) \mathrm{d}V = \int_{\Omega_e} N_i \left(-\nabla p_{\mathrm{b}} + \rho \mathbf{g}\right) \mathrm{d}V - \int_{\Gamma_e} \mathbf{n} N_i [p_{\mathrm{b}}] \mathrm{d}\Gamma \tag{10}$$

recall that $\rho \mathbf{g}$ is continuous and so integrating by parts results in no jump term. Now applying the rotation matrices this expression becomes:

$$R^{\mathrm{T}} \left(\int_{\Omega_e} R_{\mathrm{v}} N_i \left(-\nabla p_{\mathrm{b}} + \rho \mathbf{g}\right) + R_{\mathrm{h}} N_i \left(-\nabla p_{\mathrm{b}} + \rho \mathbf{g}\right) \mathrm{d}V - \int_{\Gamma_e} R_{\mathrm{v}} N_i \mathbf{n} [p_{\mathrm{b}}] + R_{\mathrm{h}} N_i \mathbf{n} [p_{\mathrm{b}}] \mathrm{d}\Gamma \right) \tag{11}$$

Once again we need to choose the discretisations to be used for the jump conditions. Consider first the vertical terms, in this case we choose the upwind scheme (6). If we further require that the function space supported by N_i is a subspace of that supported by M_i then it follows from (4) that the vertical terms cancel exactly.

In the case of the horizontal terms, there is no obvious upwind direction so the central difference jump formulation (5) is employed.

As an implementation detail, it is useful to note that:

$$R^{\mathrm{T}} R_{\mathrm{v}} = \begin{bmatrix} q_x q_x & q_x q_y & q_x q_z \\ q_y q_x & q_y q_y & q_y q_z \\ q_z q_x & q_z q_y & q_z q_z \end{bmatrix} \tag{12}$$

By applying the substition $R^{\mathrm{T}}R_{\mathrm{h}} = I - R^{\mathrm{T}}R_{\mathrm{v}}$ it is therefore possible to avoid the need to construct the tangent entries in R_{h}. The final version of the expression for the buoyancy pressure is therefore:

$$\int_{\Omega_e} (I - R^{\mathrm{T}}R_{\mathrm{v}})N_i\left(-\nabla p_{\mathrm{b}} + \rho\mathbf{g}\right)\mathrm{d}V - \int_{\Gamma_e} (I - R^{\mathrm{T}}R_{\mathrm{v}})N_i\mathbf{n}\frac{1}{2}(p_{\mathrm{b}}^{\mathrm{int}} - p_{\mathrm{b}}^{\mathrm{ext}})\mathrm{d}\Gamma \tag{13}$$